PhD Comprehensive Exams

University of North Carolina

Fall 2009

Analysis

1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Assume that

$$f^{-1}(K) \text{ is compact for all compact } K \subset \mathbb{R}^n. \tag{1}$$

   Prove that the image of f is closed.

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Assume the $n \times n$ matrix $Df(x)$ is invertible for all $x \in \mathbb{R}^n$, and that (**??**) holds. Prove that f is onto, that is, for each $y \in \mathbb{R}^n$, $y = f(x)$ for some $x \in \mathbb{R}^n$.

3. Let f be a C^1 function on $[0, 1]$ and assume

$$\int_0^1 |f'(s)|^2 \, \mathrm{d}s \leq A.$$

   Show that, for all $x, y \in [0, 1]$,

$$|f(x) - f(y)| \leq A^{1/2}|x - y|^{1/2}.$$

4. Give an example of a function f on $\mathbb{R}$ with the property that, for all $p \in (1, \infty)$,

$$\int_{-\infty}^{\infty} |f(x)|^p \, \mathrm{d}x < \infty \quad \Longleftrightarrow \quad p = 4.$$

5. Let f be an entire function with the property that, for some $A \in (0, \infty)$,

$$|f(z)| \leq A|z|, \quad \text{for all } z \in \mathbb{C}.$$

   Show that $f(z) = az$ for some $a \in \mathbb{C}$.

6. Let f be an analytic function on $\mathbb{C} \setminus 0$. Consider

$$\mathcal{A} = \left\{ z \in \mathbb{C} : \frac{1}{2} \leq |z| \leq 2 \right\}.$$

   Assume that

$$\operatorname{Im} f(z) < -1 \text{ for all } z \in \partial\mathcal{A}.$$

   Show that

$$f(1) \neq 0.$$

7. Given $\xi \in \mathbb{R}$, compute

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1 + x^2} \, \mathrm{d}x.$$

8. Construct a conformal diffeomorphism of the region

$$\mathcal{O} = \{z \in \mathbb{C} : |z| < 1,\ \text{Im } z > 0\}$$

onto the right half plane

$$\mathcal{H} = \{z \in \mathbb{C} :\ \text{Re } z > 0\}.$$

You can write it as a composition of simpler maps.

Geometry and Topology

1. Let l be a line in $\mathbb{R}^3$. Let $\alpha(s)$ be a unit speed curve in $\mathbb{R}^3$ such that its curvature is nowhere zero, its tangent line always intersects l and the curve itself does not intersect l. Prove that $\alpha(s)$ is planar.

2. Let Σ be a surface in $\mathbb{R}^3$, let $\alpha(s)$ be a unit speed curve on a surface Σ and let U be a unit normal vector field on Σ. Prove that $\alpha(s)$ is principal if and only if the parametric surface

$$\mathbf{x}(s,t) = \alpha(s) + tU(\alpha(s))$$

has zero Gaussian curvature.

3. Let Σ be a surface with a unit normal vector field U and the corresponding Gauss map $G : \Sigma \to S^2$. Suppose that the Gaussian curvature K is identically zero on Σ, and Σ has no umbilical points. Prove that for every point $p \in \Sigma$ there exists a unit speed curve $\alpha : (-\epsilon, \epsilon) \to \Sigma$ where $\epsilon > 0$ such that G is constant on α.

4. Let E_1, E_2 be an orthonormal frame field on an open set O on a surface Σ. Let $\alpha : [a,b] \to O$ be a smooth closed unit speed curve on O bounding a region D. Let $\phi(s)$ be a smooth function such that

$$\alpha' = \cos\phi\ E_1(\alpha) + \sin\phi\ E_2(\alpha).$$

(a) Define the geodesic curvature κ_g of α and prove that

$$\kappa_g(s) = \phi'(s) + \omega_{12}(\alpha'(s)).$$

(b) Show that the following expression

$$\int_a^b \kappa_g(s)\ \mathrm{d}s + \iint_D K\ \mathrm{d}A,$$

where K is the Gaussian curvature and $\mathrm{d}A$ is the volume form on Σ, is an integer multiple of 2π.

5. For $v, p \in \mathbb{R}^2$, let $v_p \in T_p\mathbb{R}^2$ be the image of v under the standard identification of the tangent space $T_p\mathbb{R}^2$ with $\mathbb{R}^2$. Define a Riemannian metric on $\mathbb{R}^2$ by the formula

$$\langle v_p, w_p \rangle = e^{2f(p)} v \cdot w,$$

where f is a smooth function on $\mathbb{R}^2$. Compute the Gaussian curvature of this Riemannian manifold.

6. The boundary of a solid torus $D^2 \times S^1$ is a torus $S^1 \times S^1$. Let X_1 and X_2 be copies of the solid torus $D^2 \times S^1$. Let X be the space obtained by identifying each point (x,y) in the boundary $S^1 \times S^1$ of X_1 with the point (y,x) in the boundary of X_2. Compute the fundamental group $\pi_1(X)$.

7. State whether the following are true or false (with brief reasons):

   (a) Let $\{Y_\alpha\}_{\alpha \in A}$ be a collection of compact subsets of a Hausdorff topological space X. Then, the intersection
   $$\bigcap_{\alpha \in A} Y_\alpha$$
   is compact.

   (b) The manifolds $S^2 \times \mathbb{R}$ and $\mathbb{R}^3 \setminus \{(0,0,0)\}$ are homeomorphic.

   (c) The group $\mathbb{C}^* := \mathbb{C} \setminus 0$ acts on the topological space $\mathbb{C}$ by multiplication. Is the quotient space $\mathbb{C}/\mathbb{C}^*$ Hausdorff?

8. (a) By any method identify the surface whose plane model is represented by the word *abcdadbc*.

   (b) Compute the fundamental group of the surface in part (a).

   (c) Find the dimension of the first de Rham cohomology group H^1 (also known as the *first Betti number*) of the surface in part (a).

9. Show that if a locally path-connected space X has a finite fundamental group, then every map $X \to S^1$ is null homotopic.

10. For each of the following spaces X, determine whether the following statement is true:

If ω is a closed 1-form on X and γ is a smooth loop in X, then the integral $\int_\gamma \omega$ must be zero.

    (a) $X = S^2$

    (b) $X = \mathbb{R}P^2$

    (c) $X = \mathbb{R}^2 \setminus \{(0,0)\}$

    Provide a brief explanation for each part.