Convergence of a Sequence

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Statement

Determine whether or not the following sequence converges or diverges:

$$a_n = \frac{(-3)^n}{n!}.$$

Suggested Solution

The easiest way to determine whether a sequence (a_n) converges or not is to evaluate its limit as n tends to infinity. As long as the limit is finite, we say that the sequence converges. However, $\lim a_n$ is not a direct computation for this particular sequence. Resorting to the L'Hôpital's Rule immediately poses another obstacle – how do we differentiate a factorial function?[1]

We shall now attempt to show that the sequence is monotonic and bounded, as this will in turn imply that our sequence is convergent.

$$|a_n| = \left| \frac{\overbrace{(-3)(-3)\cdots(-3)}^{n \text{ times}}}{n(n-1)\cdots(3)(2)(1)} \right| = \frac{3}{n} \cdot \frac{3}{n-1} \cdots \frac{3}{3} \cdot \frac{3}{2} \cdot \frac{3}{1} \leq \frac{9}{4}.$$

This inequality holds as the first $n-2$ terms are each less than or equal to 1. Clearly our sequence is bounded. However, trying to establish the monotonicity of a_n will soon appear to be futile:

$$a_{n+1} = -\frac{3}{n+1} a_n \leq -\frac{3}{2} a_n \qquad \text{for } n \geq 1.$$

It turns out that we will have to use the "direct" method *indirectly*, i.e. somehow we compute its limiting value. Observe that

$$|a_{n+1}| = \left| \frac{-3}{n+1} \right| \cdot |a_n|,$$

whence we conclude that

$$\lim_{n\to\infty} |a_{n+1}| = \lim_{n\to\infty} \left| \frac{-3}{n+1} \right| \cdot |a_n| = 0.$$

In particular, $\lim |a_{n+1}| = 0$ implies that $\lim |a_n| = 0$. Consequently, $\lim a_n = 0$ and thus our sequence converges.

[1] This certainly can be done if we consider Sterling's Theorem or the gamma function – but let's keep that out of our picture since this is an elementary calculus problem.