Math 653 – Introductory Analysis

Homework 1

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Problem 1

Let X be a metric space. If $x \in X$ and $E \subset X$, define the distance from x to E as

$$\rho(x, E) = \inf\{\rho(x, y) : y \in E\}.$$

(a) Show that the function $x \mapsto \rho(x, E)$ is uniformly continuous on X, for each $E \subset X$, and that

$$\overline{E} = \{x : \rho(x, E) = 0\}.$$

(b) Deduce from (a) that if F is a closed subset of X, there exists open sets G_n, $n \in \mathbb{N}$, such that

$$F = \bigcap_{n=1}^{\infty} G_n.$$

(c) If E and F are disjoint closed subsets of X, show that f, defined by

$$f(x) = \frac{\rho(x, E)}{\rho(x, E) + \rho(x, F)},$$

is a continuous real-valued function on X, with $0 \leq f \leq 1$, and $E = f^{-1}(\{0\})$, $F = f^{-1}(\{1\})$. Deduce that there exists disjoint open sets U and V with $E \subset U$ and $F \subset V$.

Solution

(a) Denote $f(x) = \rho(x, E)$. Using the triangle inequality, we have $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. Taking infimum on both sides, we obtain $f(x) \leq \rho(x, z) + f(z)$, or after rearranging, $f(x) - f(z) \leq \rho(x, z)$. By symmetry, we have $f(z) - f(x) \leq \rho(x, z)$ and hence $|f(x) - f(x)| \leq \rho(x, z)$.

Now pick $\epsilon = \delta$. There exists $\delta > 0$ such that $x, z \in X$ and $\rho(x, z) < \delta$ implies

$$|f(x) - f(z)| \leq \rho(x, z) < \delta = \epsilon.$$

The function $x \mapsto \rho(x, E)$ is thus uniformly continuous on X.

Recall that closure of a set E can be written as $\overline{E} = \{x : E \cap U_x \neq \emptyset\}$, where U_x denotes open sets containing x. Assume that $\rho(x, y) \neq 0$. Then there exists a number α such that $0 \leq \alpha < \rho(x, y)$ implies that $E \cap B(x, \alpha) = \emptyset$, forcing $x \notin \overline{E}$. Thus, the closure of E can be expressed as

$$\overline{E} = \{x : \rho(x, E) = 0\}.$$

(b) Since F is closed, $F = \overline{F}$ and by part (a), $F = \{x : \rho(x, F) = 0\} = \rho^{-1}(\{0\})$. Note that $\{0\} = \cap(-1/n, 1/n)$. Hence

$$F = \rho^{-1}\left[\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)\right] = \bigcap_{n=1}^{\infty}\left[\rho^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right)\right] = \bigcap_{n=1}^{\infty} G_n.$$

Here, we let $G_n = \rho^{-1}(-1/n, 1/n) \in X$. The second equality is valid due to the continuity of ρ.

(c) By part (a), we know that ρ is a continuous function. In particular, the sum of two continuous functions is continuous, and so is the quotient of two continuous function, given that the denominator is nonzero. Thus, $f(x)$ is continuous on X. Furthermore, by definition, $\rho \geq 0$ and since E and F are disjoint, $f(x)$ is a well defined, real-valued function. $0 \leq f \leq 1$ follows from the fact that $\rho(x, E) + \rho(x, F) \geq \rho(x, E)$ always, with first equality if an only if $\rho(x, E) = 0$, and with second equality if and only if $\rho(x, F) = 1$. Consequently $E = f^{-1}(\{0\})$ and $F = f^{-1}(\{1\})$. Finally, since $\mathbb{R}$ is Hausdorff, there exist open intervals I and J such that $0 \in I$, $1 \in J$ and $I \cap J = \emptyset$. Take $I = (-1/2, 1/2)$ and $J = (1/2, 3/2)$. Then it follows that $U = f^{-1}(I) \supset E$ and $V = f^{-1}(J) \supset F$. U and V are disjoint because

$$U \cap V = f^{-1}(I) \cap f^{-1}(J) = f^{-1}(I \cap J) = f^{-1}(\emptyset) = \emptyset.$$

Problem 2

Let X be a compact Hausdorff space, and let $f : X \to \mathbb{R}$. The *graph of* f is the set $G(f) = \{(x, f(x)) : x \in X\} \subset X \times \mathbb{R}$. Show that f is continuous if and only if $G(f)$ is compact.

Solution

For starter, define a function $F : X \to X \times \mathbb{R}$, mapping $x \mapsto (x, f(x))$. Let π_1 and π_2 be projection maps such that $\pi_1 \circ F = \mathbb{I}_x$ and $\pi_2 \circ F = f$, where $\mathbb{I}_x$ is the identity map of x. Since F is continuous if and only if $\pi_1 \circ F$ and $\pi_2 \circ F$ are continuous, then F is continuous if and only if f is continuous, as the identity map $\mathbb{I}_x$ is continuous.

Now, assume that f is continuous. Then F is continuous and $F(X)$ is compact since X is a compact space. But $F(x) = G(f)$. This proves that $G(f)$ is compact.

Conversely, we assume that $G(f)$ is compact. Let $H \subset G(f)$ be closed. H is compact because a closed subset of a compact space is compact. By the definition of continuity, it suffices to show that $F^{-1}(H)$ is closed to establish the continuity of F. Observe that $F^{-1}(H) = \pi_1(H)$. Since π_1 is continuous and H is compact, this implies that $\pi_1(H)$ is compact, so is $F^{-1}(H)$. Finally, $F^{-1}(H)$ is closed because a compact subset of Hausdorff space is closed. We conclude that F is continuous, and so is f, as desired.

Problem 3

Let X be a compact metric space, and $f : X \to X$ an isometry, that is $\rho(f(x), f(y)) = \rho(x, y)$ for every $x, y \in X$. Show that f is bijective.

Solution

Suppose $f(x) = f(y)$. Then $\rho(f(x), f(y)) = \rho(f(x), f(x)) = 0$. Hence $\rho(f(x), f(y)) = \rho(x, y) = 0$ implies that $x = y$. Thus f is injective.

Let $Y = f(X)$. Since f is continuous and X is compact and Hausdorff, Y is closed. Suppose f is not surjective. Then $X \setminus Y \neq \emptyset$. There exists $x_0 \in X \setminus Y$ such that $\rho(x_0, Y) = \delta > 0$, since $x_0 \notin X \setminus Y$. Define the sequence (x_n) inductively by $x_{n+1} = f(x_n)$ for every $n \geq 0$. We shall show that $\rho(x_n, x_m) \geq \delta$ for all $m < n$ by induction. For the base case,

$$\rho(x_1, x_0) \geq \rho(x_0, Y) = \delta.$$

Now, assume that for some m and n, $\rho(x_n, x_m)$ holds. Using this induction hypothesis,

$$\rho(x_{n+1}, x_{m+1} = \rho(f(x_n), f(x_m)) = \rho(x_n, x_m) \geq \delta.$$

Therefore the assumption is incorrect and f is hence surjective.

Problem 4 – Analysis Comprehensive Exam (Fall 2007)

Let $B = \{x \in \mathbb{R} : |x| \leq 1\}$ and let $C^1(B)$ be the complete normed space consisting of the continuously differentiable functions on B with the usual norm, namely,

$$||f||_1 := \sup_{x \in B} (|f(x)| + |f'(x)|) .$$

Is the closed unit ball in $C^1(B)$ compact? Prove your answer.

Solution

No. Consider the sequence of functions

$$f_n(x) = \frac{1}{3}x^{1+\frac{1}{2n-1}}.$$

f_n converges pointwise to $|x|/3$. To show that f_n lies within the closed unit ball $U \subset C^1(B)$, we compute

$$||f_n||_1 := \sup_{x\in B}\left(|f_n(x)| + |f_n'(x)|\right) = \sup_{x\in B}\left[\frac{1}{3}\left|x^{1+\frac{1}{2n-1}}\right| + \frac{1}{3}\left(1+\frac{1}{2n-1}\right)\left|x^{\frac{1}{2n-1}}\right|\right] \le \frac{1}{3}+\frac{2}{3} = 1.$$

Suppose that U is compact. Then there exists a subsequence (f_{n_k}) such that $(f_{n_k}) \to f \subset U$ in $||\cdot||_1$. However, $f = |x|/3 \notin C^1(B)$, which is a contraction. Hence the closed unit ball $U \subset C^1(B)$ is not compact.

Problem 5 – Analysis Comprehensive Exam (Fall 2007)

Suppose $g : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and that $\{f_n\}$ is a sequence of real-valued functions on [0,1]. Show that $f_n \to g$ uniformly on $[0,1]$ if and only if for every convergent sequence $\{x_n\}$ on $[0,1]$, we have

$$\lim_{n\to\infty} f_n(x_n) = g\left(\lim_{n\to\infty} x_n\right).$$

Solution

Assume that f_n converges uniformly to g. Then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$|f_n(x) - g(x)| < \epsilon \quad \text{for all } x \in [0,1].$$

By continuity of g, we may write

$$g\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} g(x_n).$$

Hence, we want to show that $\lim f_n(x_n) - g(x_n) = 0$, or equivalently,

$$|f_n(x_n) - g(x_n)| < \epsilon.$$

Since the uniform convergence of f_n guarantees $|f_n(x) - g(x)| < \epsilon$ for all $x \in [0,1]$, in particular the inequality that we sought is inherently satisfied.

We shall prove the converse by contrapositive. That is, we assume that f_n does not converge uniformly to g. This means that there exists $\epsilon > 0$ such that for all n, there exists $n > N$ and there exists x_n such that

$$|f_n(x_n) - g(x_n)| \ge \epsilon.$$

Furthermore, there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that

$$|f_{n_k}(x_{n_k}) - g(x_{n_k})| \ge \epsilon.$$

Since the interval $[0,1]$ is compact, there exists a sub-subsequence $(x_{n_{k_j}})_{j=1}^{\infty}$ that converges to $x \in [0,1]$. Finally, $|f_n(x_n) - g(x_n)| \ge \epsilon$ implies that

$$\lim_{n\to\infty} f_n(x_n) \ne g\left(\lim_{n\to\infty} x_n\right).$$

Problem 6 – Analysis Comprehensive Exam (Spring 2005)

Consider the sequence

$$a_n = \log n - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

Determine whether this sequence converges and justify your answer.

Solution

We way express a_n as

$$a_n = \log n - \sum_{i=2}^{n} \frac{1}{i} = \int_1^n \frac{1}{x}\ \mathrm{d}x - \sum_{i=2}^{n} \frac{1}{i} = \sum_{i=2}^{n} \int_{i-1}^{i} \frac{1}{x}\ \mathrm{d}x - \sum_{i=2}^{n} \frac{1}{i} = \sum_{i=2}^{n} \left[\int_{i-1}^{i} \left(\frac{1}{x} - \frac{1}{i} \right)\ \mathrm{d}x \right].$$

Note that $i - 1 \leq x \leq i$, or $1/i \leq 1/x \leq 1/(i-1)$. It follows that

$$a_n = \sum_{i=2}^{n} \left[\int_{i-1}^{i} \left(\frac{1}{x} - \frac{1}{i} \right)\ \mathrm{d}x \right] \leq \sum_{i=2}^{n} \left[\int_{i-1}^{i} \frac{1}{i(i-1)}\ \mathrm{d}x \right] = \sum_{i=2}^{n} \frac{1}{i(i-1)},$$

which is a convergent p-series. Hence a_n converges by the comparison test.

Alternatively, we may show that a_n is monotonic and bounded to prove that it is convergent. To show that it is monotonic, observe that

$$a_{n+1} - a_n = \log \frac{n+1}{n} - \frac{1}{n+1} = \int_n^{n+1} \frac{1}{x}\ \mathrm{d}x - \frac{1}{n+1} > 0.$$

Note that the definite integral represents the area under the curve $f(x) = 1/x$ from $x = n$ to $x = n+1$ while $1/(n+1)$ is the lower estimate of the rectangular area under the curve $f(x) = 1/x$ from $x = n$ to $x = n+1$.

In a similar fashion, we remark that the Riemann integral is always bounded by its lower estimate and upper estimate, that is

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \int_1^n \frac{1}{x}\ \mathrm{d}x \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}.$$

Rearranging the inequality, we have

$$0 \leq a_n \leq 1 - \frac{1}{n}.$$

Therefore, a_n is bounded from above by 1. Clearly, a_n is a convergent sequence.

Problem 7 – Analysis Comprehensive Exam (Spring 2003)

Prove that the sum

$$S(x) = \sum_{n=0}^{\infty} \frac{x^n}{n^2 x + 1}$$

converges for every $x \in [0, 1]$. Is $S(x)$ continuous on $[0, 1]$? Prove your answer.

Solution

Suppose (M_k) is a sequence of nonnegative real numbers defined as

$$M_k = \frac{x^{k-1}}{(k-1)!}.$$

Observe that $\sum M_k = e^x < \infty$ for $x \in [0, 1]$ and

$$g_k = \frac{x^k}{k^2 x + 1} \leq \frac{x^k}{(k-1)!x} = M_k,$$

then, by Weierstrass M-test, we conclude that the sum

$$S(x) = \sum_{n=0}^{\infty} \frac{x^n}{n^2 x + 1}$$

converges uniformly for every $x \in [0,1]$. To establish the continuity of $S(x)$ on $[0,1]$, it suffices to show that $g_k(x)$ is continuous on $[0,1]$. Let $\epsilon = 2/k^2$. There exists $\delta > 0$ such that $x, y \in [0,1]$ and $|x-y| < \delta$ implies

$$|g_k(x) - g_k(y)| = \left|\frac{x^k}{k^2x+1} - \frac{y^k}{k^2y+1}\right| \leq \left|\frac{x^k}{k^2x+1}\right| + \left|\frac{y^k}{k^2y+1}\right| \leq \frac{x^k}{k^2x} + \frac{y^k}{k^2y} = \frac{1}{k^2}(x^{k-1} + y^{k-1}) \leq \frac{2}{k^2} = \epsilon.$$

Hence, $g_k(x)$ is uniformly continuous on $[0,1]$, implying that it is continuous on $[0,1]$. Since g_k is continuous on $[0,1]$ and $\sum g_k$ is uniformly continuous on $[0,1]$, we therefore conclude that $S(x) = \sum g_k$ is continuous on $[0,1]$.